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A Note on Open Strings in the Presence of Constant B-Field

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Abstract

We consider the open string σ-model in the presence of a constant Neveu-Schwarz B-field on the world-sheet that is topologically equivalent to a disk with n holes. First, we compute the σ-model partition function. Second, we make a consistency check of ideas about the appearance of noncommutative geometry within open strings.
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The string σ-models provide a basic tool to derive stringy low energy effective actions that contain all orders of the fields. One example is the dilaton dependence of the effective action [1, 2, 3]: the dilaton field is coupled to the Euler characteristic χ of the corresponding string world-sheet [3] so that each world-sheet contributes a factor $\mathrm{e}^{-\chi\varphi}$ into the effective action. The second one is the Born-Infeld (BI) action which was derived using the open string σ-model on the disk [4, 5] and adopting standard renormalization schemes.[1] Later it was realized that this action plays a crucial role in D-brane physics [7, 8, 9]. It is also worth mentioning that there are known examples of string partition functions (one loop corrections to effective actions) for toroidal compactifications in the presence of constant metric and antisymmetric tensor which contain all orders of the fields [10].

Recently, it was pointed out [11] that a special renormalization scheme within the open string σ-model, a point splitting regularization, results in a rather peculiar situation where the space-time (brane) coordinates do not commute (see [11] and refs. therein). As a result, the low energy affective action becomes a noncommutative BI action. Since different renormalization schemes should be equivalent, there exists [11] a change of variables (σ-model couplings) that relates the two BI actions. The aim of this note is to check that the Seiberg-Witten relations are consistent at higher genus topologies.

We study the open string σ-model with a constant background metric and a constant Neveu-Schwarz 2-form field on the world-sheet that is topologically equivalent to a disk with n holes. Such topologies appear in the perturbation theory of open orientable strings. Thus the world-sheet action is given by

$$S=\frac{1}{4\pi\alpha'}\int_{\Sigma_n} d^2z\,\left(g_{ij}\partial_a X^i\partial^a X^j-2i\pi\alpha' B_{ij}\varepsilon^{ab}\partial_a X^i\partial_b X^j\right)+\chi\varphi \quad , \tag{1}$$

where Σ_n means the disk with n holes. g_{ij}, B_{ij}, φ are the constant metric, antisymmetric tensor and dilaton fields, respectively. X^i map the world-sheet to the brane and $i,j=1,\ldots,p+1$. The world-sheet indices are denoted by a,b.

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[1] For a review see [6] and references therein.

A natural object to compute is the σ-model partition function

$$Z_n[\varphi, g, B] = \int [dq]_n \int \mathcal{D}X \, \mathrm{e}^{-S} \quad , \tag{2}$$

where $[dq]_n$ means the modular measure for the disk with n holes extended to arbitrary dimension. It proves irrelevant for what follows, so we do not explicitly write it down [2].

To compute the partition function (2) one can follow the approach of [4] namely, first integrate over the internal points of the world-sheet to reduce the integral to the boundaries and then split the integration variable X^i on the constant x^i and non-constant parts ξ^i. Since the action is quadratic in ξ^i the problem is simply reduced to a computation of the corresponding functional determinant. The simplest case to consider is $n = 0$ i.e., the path integral on the disk. In this case the problem is equivalent to the one solved in [4]. This is clear by rewriting the term $B_{ij}\varepsilon^{ab}\partial_a X^i \partial_b X^j$ as a boundary interaction and replacing B by F [3]. A subtle point we should mention here is due to a non-diagonal metric g. So, to get a GL(p+1) invariant answer, one must be careful with the measure of the integration (see [13]). Thus the partition function on the disk computed using the ζ-function regularization is given by

$$Z_0[\varphi, g, B] = \mathrm{e}^{-\varphi} \int [dx] \, \sqrt{\det g} \, \left[\det(1 + 2\pi\alpha' g^{-1} B)\right]^{\frac{1}{2}} \quad , \tag{3}$$

where $[dx] = \frac{d^{p+1}x}{(2\pi\alpha')^{p+1}}$. The last factor is due to the integration over ξ^i. This is clear within the perturbation theory where the B-term serves as an interaction.

Our aim now is to generalize the above result for arbitrary n. In fact, what we actually need is only a generalization for the last factor in the integrand. Let us give simple, but a little bit heuristic, arguments that lead to a desired answer. It turns out that the problem has a simple solution in the framework of the so-called sewing operation for the world-sheets. The latter is based on the idea of building surfaces by sewing together other ones. So let us begin with two disks and take a cylinder as a propagator between them. It is clear that the sewing operation produces a sphere. A crucial point here is that the partition function on the sphere does not depend on B. So, restricting ourselves to the B-dependence of the partition functions, namely $Z_{\text{sphere}}[B] \sim 1$, $Z_0[B] \sim [\det(1 + 2\pi\alpha' g^{-1} B)]^{\frac{1}{2}}$, etc., we have

$$1 \sim Z_{\text{sphere}}[B] \sim Z_0[B] \, \Pi[B] \, Z_0[B] \quad . \tag{4}$$

As a result, we find the normalization of the propagator

$$\Pi[B] \sim [\det(1 + 2\pi\alpha' g^{-1} B)]^{-1} \quad . \tag{5}$$

Now let us make a consistency check and compute the partition function on the annulus. This can be done at least in two ways. The first one is to sew its boundaries to get the torus topology. The second way is to sew it with two disks. The both ways lead to the same result

$$Z_1[B] \sim \det(1 + 2\pi\alpha' g^{-1} B) \quad . \tag{6}$$

Note that such a result was also found by direct calculation in [14]. This is clear by replacing $B \to F$ and rewriting the corresponding term in (1) as boundary interactions. To be more precise, what we found corresponds to orientable non-planar diagrams for vector fields (see also [12] where this case corresponds to the two field strengths at the two boundaries set to be opposite, $F_1 = -F_2$).

[2] See, e.g., [12] where it is written down for the annulus topology.

[3] For the sake of simplicity, we use the matrix notations here and below.

It is now straightforward to get $Z_n[B]$. It is simply

$$Z_n[B] \sim [\det(1+2\pi\alpha' g^{-1}B)]^{\frac{1+n}{2}} \quad . \tag{7}$$

A crucial point here is that this factor does not depend on moduli. Hence the partition function is given by

$$Z_n[\varphi, g, B, \alpha'] = \mathrm{e}^{-\chi\varphi} \int [dq]_n \int [dx] \sqrt{\det g} \left[\det(1+2\pi\alpha' g^{-1}B)\right]^{1-\frac{1}{2}\chi} \quad . \tag{8}$$

In above we have used the fact that the Euler characteristic χ of a planar disc surface with n holes is equal to $1-n$.

Let us now give another way to derive the above result. The use of the point splitting regularization assumes that the metric g becomes a new metric G while all dependence on B can be absorbed into the so-called star product that provides a multiplication law for other background fields. In fact, in this case the action for the kinetic term is given by [15]

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma_n} d^2z \ G_{ij}\partial_a X^i \partial^a X^j \ + \chi\hat{\varphi} \quad , \tag{9}$$

while interaction terms include the build in star products. Since there are no interaction terms in the problem at hand, the partition function should have a simple structure due to standard dependence on the dilatonic field and our definition of the measure. Thus we have

$$\hat{Z}_n[\hat{\varphi}, G, \alpha'] = \mathrm{e}^{-\chi\hat{\varphi}} \int [dq]_n \int [dx] \sqrt{\det G} \quad . \tag{10}$$

The new variables found by Seiberg and Witten in case of the disk topology are [11]

$$\hat{\varphi} = \varphi + \frac{1}{2}\ln\det\bigl(G(g+2\pi\alpha' B)^{-1}\bigr) \quad , \quad G = (g-2\pi\alpha' B)g^{-1}(g+2\pi\alpha' B) \quad . \tag{11}$$

A simple algebra shows that the partition functions (8) and (10) coincide. So, the Seiberg-Witten relations (11) hold on higher topologies too.

Finally, let us make some remarks.

(i) First, let us remark that what we found can be reinterpreted in terms of vector fields. Indeed, we can consider a set of Abelian vector fields with constant field strengths $F^{(i)}$ such that different $A^{(i)}$'s coupled to different boundaries (in other words, take $n+1$ Wilson factors as interactions). A configuration of the F's that allows to rewrite the boundary interactions as the bulk term exactly corresponds what we considered. From the physical point of view, such a configuration represents $n+1$ free Wilson factors with each factor contributing the Born-Infeld determinant.

(ii) Second, it was conjectured in [11] that there exists a suitable regularization that interpolates between the Pauli-Villars (ζ-function) regularization and the point splitting one. In the framework of the open string σ-model such a regularization was further developed in [15] where it was proposed to use the world sheet action

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma_n} d^2z \left(\tilde{G}_{ij}\partial_a X^i \partial^a X^j - 2i\pi\alpha' \Phi_{ij}\varepsilon^{ab}\partial_a X^i \partial_b X^j\right) + \chi\tilde{\varphi} \quad , \tag{12}$$

where

$$\tilde{\varphi} = \varphi + \frac{1}{2}\ln\det\Bigl(\frac{\tilde{G}+2\pi\alpha'\Phi}{g+2\pi\alpha' B}\Bigr) \quad , \quad \tilde{G} = \Bigl(G^{-1} - \frac{1}{(2\pi\alpha')^2}\theta_0 G\theta_0\Bigr)^{-1} \quad , \quad \Phi = -\frac{1}{(2\pi\alpha')^2}\tilde{G}\theta_0 G \quad . \tag{13}$$

Here θ_0 is a free matrix parameter. G is given by Eq. (11).

Repeating the analysis that led us to Eq. (8), we can easily write down the partition function in the case of interest

$$\tilde{Z}_n[\tilde{\varphi}, \tilde{G}, \Phi, \alpha'] = \mathrm{e}^{-\chi\tilde{\varphi}} \int [dq]_n \int [dx] \sqrt{\det \tilde{G}} \left[\det(1 + 2\pi\alpha' \tilde{G}^{-1}\Phi)\right]^{1-\frac{1}{2}\chi} \quad . \tag{14}$$

A simple algebra shows that $\tilde{Z}_n$ coincides with Z_n, so it also passes a consistency check on higher genus topologies.

(iii) It is not difficult to formally repeat the previous analysis for superstring. To do so, it is more convenient to consider the NSR formalism within the point splitting regularization. In other words, we add a set of the fermionic fields ψ^i whose metric also is G_{ij}. It is simply to suggest what the superstring partition function should be.

$$\hat{\mathbf{Z}}_n[\varphi, G, \alpha'] = \mathrm{e}^{-\chi\hat{\varphi}} \int [d\mathbf{q}]_n \int [dx] \sqrt{\det G} \quad , \tag{15}$$

where $[d\mathbf{q}]_n$ means a proper modular measure for superstring. Clearly, there is no problem with rewriting this expression in terms of g, B and φ.

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